Exam AL June 20, 2017

① a) 1) BASIS: All atoms p are wffs of $\mathcal{L}_{\neg,\to}$

2) IND: If P and Q are wffs of $\mathcal{L}_{\neg,\to}$, then so are $\neg P$ an $P \to Q$.

3) CLOSURE: Nothing else is a wff of $\mathcal{L}_{\neg,\to}$, unless it can be built in finitely many steps from 1, 2.

b) i)

$$p' = p \text{ for atoms } p$$
$$(P \wedge Q)' = \neg(P' \to \neg Q')$$
$$(P \vee Q)' = \neg P' \to Q'$$
$$(P \to Q)' = P' \to Q'$$
$$(\neg P)' = \neg P'$$
$$(P \leftrightarrow Q)' = (\neg Q' \to P') \to \neg(P' \to \neg Q')$$

(other equ

ii) BASIS: For atoms p, $p' = p$, which is in $\mathcal{L}_{\neg,\to}$ by clause 1 of the definition.

IH: Let P and Q be two arbitrary wffs in classical propositional logic (CPL) and suppose that both $P' \in \mathcal{L}_{\neg,\to}$ and $Q' \in \mathcal{L}_{\neg,\to}$.

IND. STEP:
- $(\neg P)' = \neg P'$. By the IH, we know $P' \in \mathcal{L}_{\neg,\to}$ and so, by clause 2 of the definition of $\mathcal{L}_{\neg,\to}$, $\neg P' \in \mathcal{L}_{\neg,\to}$. Therefore, $(\neg P)' \in \mathcal{L}_{\neg,\to}$.
- $(P \to Q)' = P' \to Q' \in \mathcal{L}_{\neg,\to}$ by the IH and def. $\mathcal{L}_{\neg,\to}$.
- $(P \wedge Q)' = \neg(P' \to \neg Q') \in \mathcal{L}_{\neg,\to}$ by the IH and repeated application of clause 2 of def. $\mathcal{L}_{\neg,\to}$.
- $(P \vee Q)' = \neg P' \to Q' \in \mathcal{L}_{\neg,\to}$ by IH and def. $\mathcal{L}_{\neg,\to}$.
- $(P \leftrightarrow Q)' = (\neg Q' \to P') \to \neg(P' \to Q')$ ~~by~~ $\in \mathcal{L}_{\neg,\to}$ by IH and def. $\mathcal{L}_{\neg,\to}$.

CONCLUSION: Therefore, for all wffs P in CPL, P' is in $\mathcal{L}_{\neg,\to}$.

Alternative versions for $(P \leftrightarrow Q)'$:
- $(\neg Q' \to P') \to \neg(P' \to \neg Q')$
- $\neg((P' \to Q') \to \neg(Q' \to P'))$
- $(P' \to \neg Q') \to \neg(\neg P' \to Q')$

1b iii We want to show that for all propositional formulas and all valuations v, $v(P') = v(P)$. Take any v.

BASIS: for atoms p, $v(p') = v(p)$ by def, because $p' = p$

IH: Suppose that for some arbitrary wffs P, Q: $v(P') = v(P)$ and $v(Q') = v(Q)$.

IND. STEP: $v((\neg P)') \overset{def}{=} v(\neg P') = 1 - v(P') \overset{IH}{=} 1 - v(P) = v(\neg P)$

- $v((P \wedge Q)') \overset{def}{=} v(\neg(P' \to \neg Q')) = \min(v(P'), v(Q')) \overset{IH}{=} \min(v(P), v(Q)) = v((P \wedge Q))$
- $v((P \vee Q)') \overset{def}{=} v(\neg P' \to Q') = \max(v(P'), v(Q') \overset{IH}{=} \max(v(P), v(Q)) = v((P \vee Q))$.
- $v((P \to Q)') \overset{def}{=} v(P' \to Q') = \max(1 - v(P'), v(Q')) \overset{IH}{=} \max(1 - v(P), v(Q)) = v(P \to Q)$

- $v((P \leftrightarrow Q)') \overset{def}{=} v((\neg Q' \to P') \to \neg(P' \to \neg Q')) =$
$$\max(\min(1 - v(Q'), 1 - v(P')), \min(v(P'), v(Q'))) \overset{IH}{=}$$
$$\max(\min(1 - v(Q), 1 - v(P)), \min(v(P), v(Q))) =$$
$$v(P \leftrightarrow Q)$$

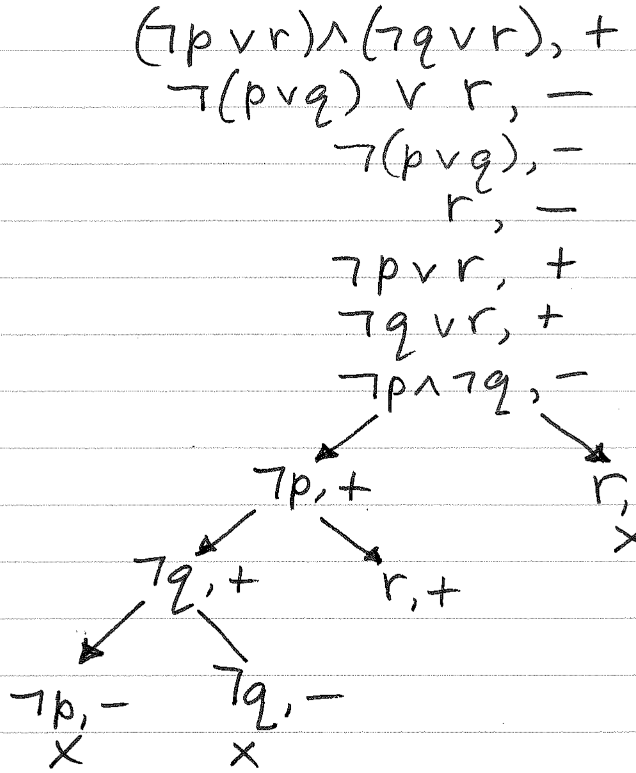
Conclusion: Therefore for all wffs P, $v(P') = v(P)$. Because v was arbitrary, this means that for all wffs P, P and P' are logically equivalent

②

	p	q	$((p \supset q) \supset p) \supset p$		
1)	1	1	1	1	1
2)	1	i	i	1	1
3)	1	0	0	1	1
4)	i	1	1	i	1
5)	i	i	1	i	1
6)	i	0	i	1	$\boxed{i}$
7)	0	1	1	0	1
8)	0	i	1	0	1
9)	0	0	1	0	1

In $Ł_3$, we have $D = \{1\}$. The valuation of row 6) provides a countermodel: for $v(p) = i$ an $v(q) = 0$, $v((p \supset q) \supset p) \supset p) = i \notin D$, so the inference does <u>not</u> hold in $Ł_3$.

③



All branches close, so the tableau is closed, so the inference $(\neg p \vee r) \wedge (\neg q \vee r) \vdash_{LP} \neg(p \vee q) \vee r$ is valid

④ For $D = \{x : x \geq 0.9\}$, the following inference <u>is</u> valid: $(p \to q) \to q \models_{0.9} p \vee q$,

that is, for all valuations v:
if $v((p \to q) \to q) \geq 0.9$, then
$v(p \vee q) \geq 0.9$.

We show this by cases:

- If $v(p) \leq v(q)$, then $v(p \to q) = 1$ and
  $v((p \to q) \to q) = 1 - (1 - v(q)) = v(q)$
  $\leq \max(v(p), v(q)) =$
  $v(p \vee q)$
- If $v(p) > v(q)$, then $v(p \to q) = 1 - v(p) + v(q)$
  $> v(q)$.
  So $v((p \to q) \to q) = 1 - (1 - v(p) + v(q)) + v(q)$
  $= v(p) \leq \max(v(p), v(q)) =$
  $v(p \vee q)$.

So in both cases, if $v((p \to q) \to q) \geq 0.9$, then $v(p \vee q) \geq 0.9$.

⑤

$\Diamond(\Diamond p \wedge \Diamond q), 0$

$\neg(\Diamond\Diamond p \wedge \Diamond\Diamond q), 0$

$0 r 1$

$\Diamond p \wedge \Diamond q, 1$

$\Diamond p, 1$

$\Diamond q, 1$

$1 r 2$

$p, 2$

$1 r 3$

$q, 3$

$\neg\Diamond\Diamond p, 0$	$\neg\Diamond\Diamond q, 0$
$\neg\Diamond p, 1$ (← or close here)	$\neg\Diamond q, 1$ (← or close here)
$\neg p, 2$	$\neg q, 2$
×	×

All branches are closed so the tableau is closed, so the inference $\Diamond(\Diamond p \wedge \Diamond q) \vdash_K \Diamond\Diamond p \wedge \Diamond\Diamond q$ is valid.

⑥ $\langle F\rangle[P]q\ ,\ 0$

$\neg[F][P]q\ ,\ 0$

$0r1$

$[P]q\ ,\ 1$

$q\ ,\ 0$

$0r2$

$\neg[P]q\ ,\ 2$

$3r2$

$\neg q\ ,\ 3$

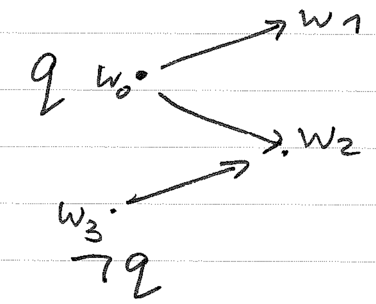
$\uparrow$

open complete branch.

The inference $\langle F\rangle[P]q \vdash_{K^t} [F][P]q$ is not valid, because there is ~~an~~ open complete branch, which gives rise to a countermodel: $I=\langle W,R,v\rangle$ with $W=\{w_0, w_1, w_2, w_3\}$,

$R=\{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle, \langle w_3, w_2\rangle\}$ and

$v_{w_0}(q)=1$, $v_{w_3}(q)=0$, and $\left.\begin{matrix} v_{w_2}(q)=0/1 \\ v_{w_1}(q)=0/1 \end{matrix}\right\}$ all allowed



⑦ Let b be a complete open branch of a K_η-tableau, and let $I = \langle W, R, v \rangle$ be the interpretation that is induced by b.

This means: $W = \{ w_i : i \text{ occurs on } b \}$

$R = \{ \langle w_i, w_j \rangle : irj \text{ is on } b \}$

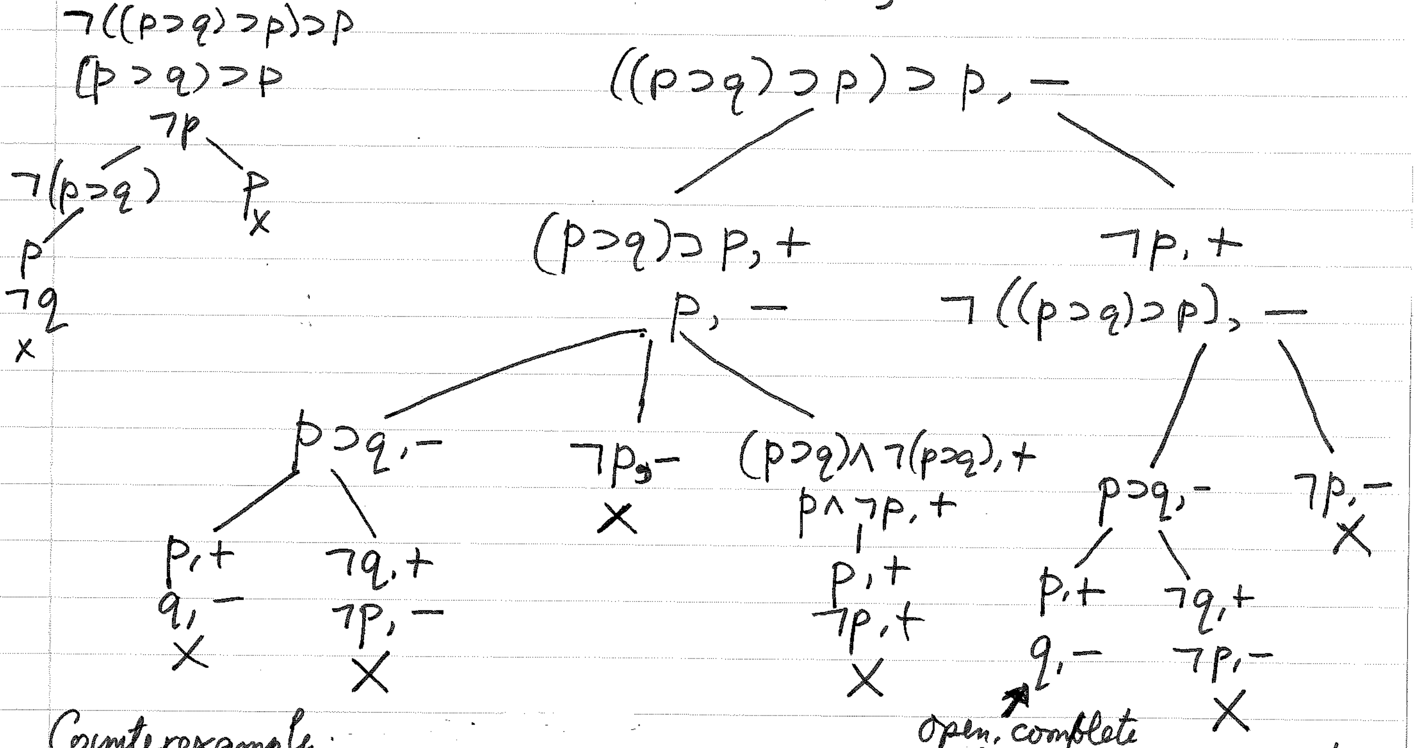
for atoms p: $\begin{cases} v_{w_i}(p) = 1 & \text{if } p, i \text{ is on } b \\ v_{w_i}(p) = 0 & \text{if } \neg p, i \text{ is on } b \end{cases}$

We want to show that R is extendable. So suppose that $w \in W$, then $w = w_i$ for some i occurring on b.

Because b is a complete open branch of a K_η-tableau, η has been applied, so irj occurs on the branch (for some new j). So $w_i R w_j$ for that j.

Therefore, for all $w_i \in W$, there is a $w_j \in W$ such that $w_i R w_j$, i.e, R is extendable.

⑩ For the formula $((p \supset q) \supset p) \supset p$, we have $\vdash ((p \supset q) \supset p) \supset p$, but not $\vdash_{RM_3} ((p \supset q) \supset p) \supset p$



Counterexample:

$p\rho 0$, $p\rho 1$, $q\rho 0$ This gives $((p \supset q) \supset p) \supset p)\rho 0$! and not $((p \supset q) \supset p) \supset p)\rho 1$

⑨ a) (i) ( ) is a process, because for all defaults in it (there are none), 'δ_k is applicable to $\Pi[k]$' in an empty way.

( ) is not closed, because δ_2 and δ_3 can be applied to it

( ) is successful, because $Out() = \emptyset$ so $In() \cap Out() =$

ii) (δ_2) is a process because δ_2 is applicable to $\emptyset$
$In() = Th\{C(d), \neg C(d) \vee A(d)\}$. This is because the prerequisite $C(d) \in In()$, while $\underbrace{\neg A(d)}_{\text{justification}} \notin In()$.

(δ_2) is not closed, because δ_1 and δ_3 can be applied to it

(δ_2) is successful, because $In(\delta_2) = Th(W \cup H(d)) = Th(C(d), A(d), H(d))$ and $Out(\delta_2) = \{\neg A(d)\}$, so $In(\delta_2) \cap Out(\delta_2) = \emptyset$

(iii) (δ_2, δ_3) is a process because of (ii) and δ_3 is applicable to $In(\delta_2) = Th(C(d), A(d), H(d))$. This is because the prerequisite $A(d) \in In(\delta_2)$, while the negation of the justification isn't: $\neg C(d) \notin In(\delta_2)$.

(δ_2, δ_3) is closed, because δ_1 is not applicable to $In(\delta_2, \delta_3) = Th(C(d), A(d), H(d), \neg F(d))$: the negation of the justification, namely $\neg F(d)$, $\in In(\delta_2, \delta_3)$.

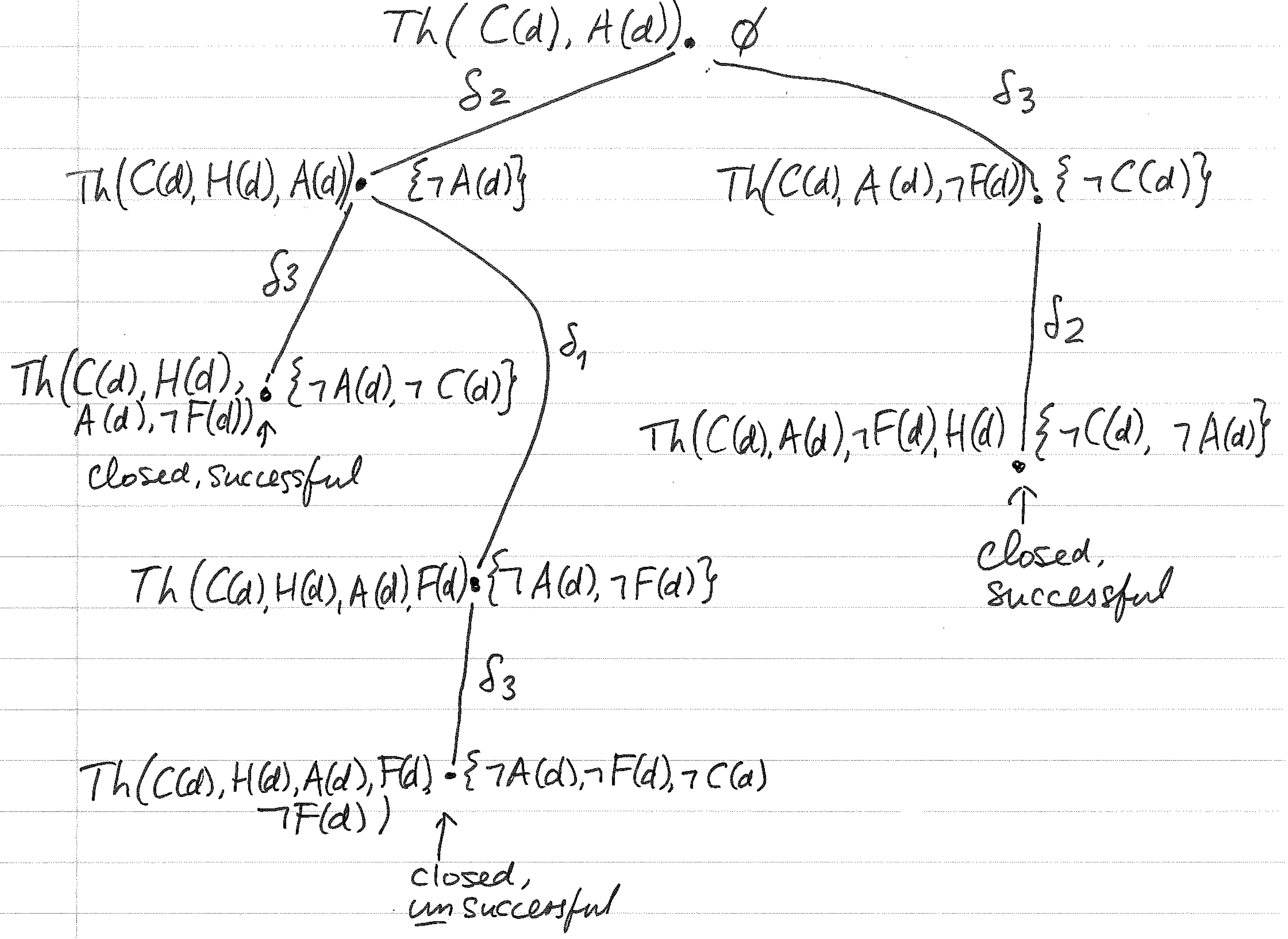
(δ_2, δ_3) is successful because $Out(\delta_2, \delta_3) = \{\neg A(d), \neg C(d)\}$, so $In(\delta_2, \delta_3) \cap Out(\delta_2, \delta_3) = \emptyset$

IV) $(\delta_2, \delta_3, \delta_1)$ is <u>not</u> a process, because δ_1 is not applicable to $In(\delta_2, \delta_3)$, See (iii).

9 b) Note that $Th(W) = Th(\{C(d), \neg C(d) \vee A(d)\}) = Th(\{C(d), A(d)\})$

In the tree below, to save space, I leave out the curly brackets in $Th(\{\ldots\})$.

Process tree :



c) There is one extension, corresponding to $In(\delta_2, \delta_3) = In(\delta_3, \delta_2) = Th(\{C(d), A(d), \neg F(d), H(d)\})$

⑧

$$\exists x \Diamond (Px \land Qx), 0$$
$$\neg (\Diamond \exists x Px \land \Diamond \exists x Qx), 0$$
$$\mathcal{E} a, 0$$
$$\Diamond (Pa \land Qa), 0$$
$$0r1$$
$$Pa \land Qa, 1$$
$$Pa, 1$$
$$Qa, 1$$

$$\neg \Diamond \exists x Px, 0 \qquad \neg \Diamond \exists x Qx, 0$$
$$\neg \exists x Px, 1 \qquad \neg \exists x Qx, 1$$
$$\forall x \neg Px, 1 \qquad \forall x \neg Qx, 1$$
$$\neg \mathcal{E} a, 1 \quad \neg Pa, 1 \qquad \neg \mathcal{E} a, 1 \quad \neg Qa, 1$$

open, complete × open, complete ×

There are two open, complete branches so the tableau is open and the inference $\exists x \Diamond (Px \land Qx) \vdash_{VK} \Diamond \exists x Px \land \Diamond \exists x Qx$ is not valid.

Both open branches lead to the same countermodel $I = \langle D, W, R, v \rangle$:

$D = \{\delta_a\}$ $D_{w_0} = \{\delta_a\}$
$W = \{w_0, w_1\}$ $D_{w_1} = \emptyset$
$R = \{\langle w_0, w_1 \rangle\}$
$v_{w_1}(P) = \{\delta_a\}$

$v_{w_1}(Q) = \{\delta_a\}$ $v_{w_0}(\mathcal{E}) = \{\delta_a\}$
$v_{w_0}(P) = \emptyset \,/\, \{\delta_a\}$ $v_{w_1}(\mathcal{E}) = \emptyset$
$v_{w_0}(Q) = \emptyset \,/\, \{\delta_a\}$